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A Generalization of Theorems of Eidelheit and Carleman Concerning Approximation and Interpolation

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We prove necessary and sufficient conditions for linear operators to approximate and interpolate unbounded continuous functions on certain subsets $U \subseteq (-\infty, \infty)$. The main application of our general theory is to simultaneous asymptotic approximation and interpolation by function series. Special cases of our results are a sharpened version of a theorem of Eidelheit for the solubility of infinite systems of linear equations and a generalization of a theorem of Carleman concerning the asymptotic approximation and interpolation of continuous functions by entire functions on the real axis. Moreover we can apply our general theorems to a moment problem of Pólya and to asymptotic approximation and interpolation by Dirichlet series. Our general approach to such problems is based on the use of certain complete approximation systems and on an essential identity theorem of functional analysis concerning approximations in normed linear spaces with certain additional restrictions by seminorms.

1. INTRODUCTION

In this paper we give an account of a more general theory concerning the simultaneous asymptotic approximation and interpolation of unbounded continuous functions by linear operators.

Let X always denote a linear space. For our applications especially we choose X to be one of certain sequence spaces or function spaces. Let $C(U)$ denote the class of all complex-valued and continuous functions f on subsets $U \subseteq (-\infty, \infty)$. For each $s \in U$ let F_s be a linear operator from X into the set of complex numbers such that

$$F_s(\alpha x + \beta y) = \alpha F_s(x) + \beta F_s(y) \qquad (x, y \in X; \alpha, \beta \text{ complex}; s \in U).$$

The system (U, X, F_s) is said to have the asymptotic approximation property (A) if for every $f, h \in C(U)$, $h(s) > 0$ $(s \in U)$ there exists an element $x \in X$ such that

$$|f(s) - F_s(x)| < h(s) \qquad (s \in U). \tag{1}$$

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Replacing (1) by the inequality

$$|f(s)-F_s(x)|<\varepsilon \qquad (s\in U)$$

for any assigned $\varepsilon>0$ we define the property $(\tilde{A})$. Furthermore, the system (U, X, F_s) has the simultaneous approximation and interpolation property (A, I) if for every $f, h\in C(U)$, $h(s)>0$ $(s\in U)$, and for every sequence (q_ν) of distinct $q_\nu\in U$ $(\nu=1, 2, ...)$, where each compact subset of U only contains a finite number of these q_ν, there is an element $x\in X$ satisfying

$$|f(s)-F_s(x)|<h(s) \quad (s\in U) \qquad \text{and} \qquad F_{q_\nu}(x)=f(q_\nu) \quad (\nu=1, 2, ...).$$

The main result of this paper is the proof of necessary and sufficient conditions for our systems (U, X, F_s) to possess the properties (A), $(\tilde{A})$, (A, I) respectively.

Our main application is to the asymptotic approximation and interpolation of unbounded continuous functions by function series on the set U if we choose

$$F_s(x)=\sum_{k=1}^{\infty} a_k K_k(s)$$

with given functions $K_k\in C(U)$ $(k=1, 2, ...)$, where X is the linear space of certain sequences $x=(a_k)$ of the coefficients a_k. A special case of this application is a proof of a sharpened version of a theorem of Eidelheit [2, p. 145] concerning the solubility of infinite systems of linear equations if we choose U to contain only isolated points $t_i\in(-\infty, \infty)$ $(i=1, 2, ...)$. We also deduce from our general result for function series a generalization of a theorem of Carleman concerning the asymptotic approximation and interpolation of continuous functions on the real axis by entire functions if $U=(-\infty, \infty)$. Applying our results in the case $K_k(s)=s^{\lambda_k}$ with given exponents $\lambda_k\geqslant 0$ $(k=0, 1, 2, ...)$ we obtain necessary and sufficient conditions for the asymptotic approximation and interpolation by Dirichlet series on $[0, 1)$ or on $[0, \infty)$. This extends the well known approximation by Müntz polynomials. Finally, we strengthen a result of Pólya concerning a moment problem if we choose X to be a class of certain entire functions.

Our general theory is based on a definition of complete approximation systems (U, X, F_s) with respect to a sequence (p_n) of given seminorms p_n on X. We might mention that these complete systems are quite different from the rather complicated (locally convex) F-spaces, which are used in [2; 6, I, p. 208, and II, p. 127; 9] to prove the theorem of Eidelheit. Of course in various areas of analysis F-spaces are of greater importance for problems of sequence spaces. But by their simpler "topological nature" our complete systems are more convenient especially for applications to problems of simultaneous approximation and interpolation. All of our proofs are based on only one essential identity theorem of functional analysis concerning a relationship between approximations in normed linear spaces that have certain additional restrictions by seminorms and corresponding identity properties of bounded linear functionals on these spaces. This theorem is a consequence of the Hahn–Banach theorem and enables us to simplify our proofs such that we do not need inverse procedures such as [2, p. 140; 6].

2. Special Results

To motivate our more abstract methods, definitions, and theorems we first consider the following three special results from different parts of the theory of approximation and interpolation. These are special problems of sequence spaces and function spaces X. Then our general theory enables us to obtain improvements of these theorems as applications of a systematic whole.

First, we say that an infinite matrix (a_{ik}) (a_{ik} complex; $i, k = 1, 2, \ldots$) has the property (E) if for arbitrarily given complex numbers c_i ($i = 1, 2, \ldots$) the infinite system of linear equations

$$\sum_{k=1}^{\infty} a_{ik} x_k = c_i \qquad (i = 1, 2, \ldots)$$

has a solution $x = (x_k)$ (x_k complex) such that $\sum_{k=1}^{\infty} |a_{ik} x_k| < \infty$ ($i = 1, 2, \ldots$). The problem to determine necessary and sufficient conditions for a matrix (a_{ik}) to satisfy (E) was first completely solved by the following theorem of Eidelheit [2, p. 145], where the proof of [2] is based on the theory of (locally convex) F-spaces combined with a complicated inverse operation [2, Theorem 1, p. 140].

Theorem 1. *A matrix* (a_{ik}) $(i, k = 1, 2, \ldots)$ *has the property* (E) *if and only if the following conditions* (L) *and* (N) *are satisfied*:

(L) $\sum_{\nu=1}^{i} \lambda_\nu a_{\nu k} = 0$ $(k = 1, 2, \ldots)$ *implies* $\lambda_\nu = 0$ $(\nu = 1, \ldots, i)$ *for each* $i = 1, 2, \ldots$; *i.e., the rows of the matrix are linearly independent*;

(N) *for each* $n = 1, 2, \ldots$ *there is an integer* $i_n > n$ *such that for all* $i \geqslant i_n$ *the inequality* $|\sum_{\nu=1}^{i} \lambda_\nu a_{\nu k}| \leqslant M \sum_{\nu=1}^{n} |a_{\nu k}|$ $(k = 1, 2, \ldots)$ *implies* $\lambda_\nu = 0$ $(i_n \leqslant \nu \leqslant i)$, *where* M *is independent of* k.

For generalizations see also [2, p. 143; 6, II, p. 125; 9]. Moreover we say that a matrix (a_{ik}) has the property $(\tilde{\mathrm{E}})$ if, given complex numbers c_i $(i=1, 2, \ldots)$ and $\varepsilon > 0$, we can find $x=(x_k)$ satisfying

$$\left| c_i - \sum_{k=1}^{\infty} a_{ik} x_k \right| < \varepsilon \qquad \text{and} \qquad \sum_{k=1}^{\infty} |a_{ik} x_k| < \infty \quad (i=1, 2, \ldots).$$

Applying our general theorems we obtain a new and simplified proof of the theorem of Eidelheit and a stronger result by

THEOREM 2. *A matrix* (a_{ik}) $(i, k=1, 2, \ldots)$ *has the property* (E) *if and only if* (a_{ik}) *has the property* $(\tilde{\mathrm{E}})$, *and each of these properties is equivalent to each of the two statements* (a), (b):

(a) (L) *and* (N) *are satisfied*;

(b) (L) *and the following property are satisfied*: *for each* $n=1, 2, \ldots$ *there is an integer* $i_n > n$ *such that, given any complex numbers* c_i $(i=1, \ldots, l)$, $l \geqslant i_n$ *with* $c_i = 0$ $(1 \leqslant i < i_n)$ *and* $\varepsilon > 0$, *we can find complex numbers* x_k $(k=1, \ldots, m)$ *satisfying*

$$\left| c_i - \sum_{k=1}^{m} a_{ik} x_k \right| < \varepsilon \qquad (i=1, \ldots, l) \qquad \textit{and}$$

$$\sum_{k=1}^{m} |a_{ik} x_k| < \varepsilon \qquad (i=1, \ldots, n).$$

Concerning the second special result a theorem of Carleman [1, 3] asserts that for every $f, h \in C(-\infty, \infty)$, $h(s) > 0$, there exists an entire function g such that

$$|f(s) - g(s)| < h(s) \qquad (-\infty < s < \infty).$$

To extend this result we say that a sequence (m_k) of integers $m_k \geqslant 0$ $(k=0, 1, 2, \ldots)$ has the property (A) if for every $f, h \in C(-\infty, \infty)$, $h(s) > 0$, we can find an entire function g with $g(s) = \sum_{k=0}^{\infty} a_k s^{m_k}$ such that

$$|f(s) - g(s)| < h(s) \qquad (-\infty < s < \infty). \tag{2}$$

Moreover, if for any numbers $q_i \in (-\infty, \infty)$ $(i=1, 2, \ldots)$, $|q_i| \to \infty$ $(i \to \infty)$ we can choose $g(s) = \sum_{k=0}^{\infty} a_k s^{m_k}$ to satisfy, in addition to (2), the equations

$$g(q_i) = f(q_i) \qquad (i=1, 2, \ldots),$$

we say that (m_k) has the approximation and interpolation property (A, I). To improve the theorem of Carleman we deduce from our general theorems

THEOREM 3. *A sequence* (m_k) *of integers* m_k, $0=m_0<m_k<m_{k+1}$ $(k=1, 2, ...)$ *has the property* (A) *if and only if* (m_k) *has the property* (A, I), *and each of these properties is satisfied if and only if*

$$\sum_{\substack{k\geqslant 1\\ m_k \text{ even}}} m_k^{-1}=\infty \qquad \text{and} \qquad \sum_{\substack{k\geqslant 1\\ m_k \text{ odd}}} m_k^{-1}=\infty. \tag{3}$$

Finally, the following result concerning moment problems is due to Pólya [10]:

THEOREM 4. *For arbitrarily given complex numbers* c_i $(i=0, 1, 2, ...)$ *there is an entire function g such that*

$$\int_0^\infty |g(u)|\, u^s\, du<\infty \quad (s\geqslant 0)$$

and

$$\int_0^\infty g(u)u^i\, du=c_i \quad (i=0, 1, 2, ...).$$

Strengthening Theorem 4 we shall prove

THEOREM 5. *For every* $f, h\in C[0, \infty)$, $h(s)>0$ $(s\geqslant 0)$, $q_i\geqslant 0$ $(i=1, 2, ...)$, $q_i\to\infty$ $(i\to\infty)$ *there exists an entire function g such that*

$$\int_0^\infty |g(u)|\, u^s\, du<\infty \qquad (s\geqslant 0),$$

$$\left|f(s)-\int_0^\infty g(u)u^s\, du\right|<h(s) \qquad (s\geqslant 0),$$

and

$$\int_0^\infty g(u)u^{q_i}\, du=f(q_i) \qquad (i=1, 2, ...).$$

3. GENERAL THEOREMS

We now generalize Theorem 1 and 2 for our systems (U, X, F_s). This requires the definition of a suitable completeness of (U, X, F_s), and the linear independence of the rows in the conditions (L) and (N) of

Theorem 1 shall be replaced and generalized by analogous identity properties concerning Stieltjes integrals to represent the corresponding linear functionals.

We now assume that $U=\bigcup_{i=1}^{\infty} D_i$ with closed intervals $D_i=[a_i, b_i]$, $-\infty<a_i\leqslant b_i<\infty$ $(i=1, 2, \ldots)$. We suppose that the open intervals (a_i, b_i) are disjoint. Thus the D_i may have common endpoints. The case $a_i=b_i$ also is admitted, where in this case we assume that $D_i \cap D_j=\varnothing$ $(j\neq i)$; i.e. D_i is an isolated point if $a_i=b_i$.

If p_n $(n=1, 2, \ldots)$ are seminorms on the linear space X [12, p. 24], we say that the system (U, X, F_s) is complete with respect to (p_n) if $\sum_{i=1}^{\infty} p_i(x_i)<\infty$, $x_i \in X$ always implies (α) and (β):

$$(\alpha) \quad \sum_{i=1}^{\infty} |F_s(x_i)|<\infty \qquad (s\in U), \tag{4}$$

(β) there is an element $x\in X$ such that

$$F_s(x)=\sum_{i=1}^{\infty} F_s(x_i) \qquad (s\in U). \tag{5}$$

In the following, for the systems (U, X, F_s), we always assume that $F_s(x)$ presents a continuous function on U concerning s for each fixed $x\in X$.

We set $B_n=\bigcup_{i=1}^{n} D_i$ $(n=1, 2, \ldots)$. Generalizing condition (L) of Theorem 1 we say that the system (U, X, F_s) has the identity property (W) if for each fixed $n=1, 2, \ldots,$

$$\int_{B_n} F_t(x)\, d\alpha(t)=0 \quad (x\in X), \qquad \int_{B_n} |d\alpha(t)|<\infty$$

imply $\alpha(t)=0$ $(t\in B_n)$ for a normalized function α on B_n. If $B_n=\bigcup_{i=1}^{n} D_i=\bigcup_{j=1}^{m} A_j$ with disjoint closed intervals $A_j=[c_j, d_j]$, $c_j\leqslant d_j$, then the normalization of α means that $\alpha(t)=2^{-1}[\alpha(t+0)+\alpha(t-0)]$ for all inner points of B_n, and that $\alpha(t_e)=0$ for an endpoint t_e of A_j in the case $c_j<d_j$. Here we have $\int_{A_j} F_t(x)\, d\alpha(t)=\lambda F_{c_j}(x)$ with some constant λ if $c_j=d_j$, and the conclusion of (W), that $\alpha(t)=0$ on A_j, in this case means $\lambda=0$.

Our condition (W) is the linear independence of the operators F_{t_i} on X if U only contains isolated points t_i $(i=1, 2, \ldots)$, and this is the first condition in the more general theorem of Eidelheit [2, p. 143].

To generalize (N) of Theorem 1 we say that the system (U, X, F_s) has the property (M) with respect to a sequence (p_n) of seminorms p_n on X if for each n there is an integer $i_n>n$ such that for all $i\geqslant i_n$ the condition

$$\left|\int_{B_i} F_t(x)\, d\alpha(t)\right|\leqslant Mp_n(x) \quad (x\in X), \qquad \int_{B_i} |d\alpha(t)|<\infty \tag{6}$$

implies $\alpha(t)=0$ on all D_ν $(i_n \leqslant \nu \leqslant i)$ for a normalized α on $B_i=\bigcup_{\nu=1}^{i} D_\nu$, where M is independent of x.

Finally, we say that (U, X, F_s) has the property $(A_{p,m,j})$ concerning a seminorm p on X for positive integers m, j with $1<m\leqslant j$ if for each $f\in C(B_j)$ with $f(s)=0$ $(s\in B_{m-1})$, and for any $\varepsilon>0$, there exists an element $x\in X$ such that

$$|f(s)-F_s(x)|<\varepsilon \quad (s\in B_j) \qquad \text{and} \qquad p(x)<\varepsilon. \tag{7}$$

Moreover, if we can choose $x\in X$ to satisfy, in addition to (7), the equations $F_{q_\nu}(x)=f(q_\nu)$ $(\nu=1, ..., N)$ for arbitrarily given numbers $q_\nu\in B_j$ $(\nu=1, ..., N)$ we say that (U, X, F_s) has the property $(A^{\mathrm{I}}_{p,m,j})$ concerning p for the integers m, j.

We now state our main result:

THEOREM 6. *Suppose that the system* (U, X, F_s) *is complete with respect to the sequence of seminorms* p_n $(n=1, 2, ...)$ *on* X. *Then the properties* (A), $(\tilde{\mathrm{A}})$, (A, I) *are equivalent, and each of these properties is equivalent to each of the following statements* (a), (b), (c):

(a) (W) *and* (M) *are satisfied*;

(b) (W) *and the following property are satisfied: for each* $n=1, 2, ...$ *there exists an integer* $i_n>n$ *such that* $(A_{p_n, i_n, i})$ *is valid for all* $i\geqslant i_n$;

(c) (W) *and the following property are satisfied: for each* $n=1, 2, ...$ *there exists an integer* $i_n>n$ *such that* $(A^{\mathrm{I}}_{p_n, i_n, i})$ *is valid for all* $i\geqslant i_n$.

The main application of Theorem 6 is to asymptotic approximation and interpolation by function series, and we use in this special case the following notations: If $K=(K_k)$ is a sequence of functions K_k $(k=1, 2, ...)$ on $U=\bigcup_{i=1}^{\infty} D_i$, we define $X=L_K$ to be the set of all sequences $x=(a_k)$ (a_k complex) such that $\sum_{k=1}^{\infty} |a_k K_k(s)|$ is bounded on each D_i, and such that $\sum_{k=1}^{\infty} a_k K_k(s)$ converges uniformly on each D_i $(i=1, 2, ...)$. We take

$$F_s(x)=\sum_{k=1}^{\infty} a_k K_k(s) \qquad (x=(a_k)\in L_K, s\in U), \tag{8}$$

and the seminorms

$$p_n(x)=\sup_{s\in B_n} \sum_{k=1}^{\infty} |a_k K_k(s)| \qquad (x=(a_k)\in L_K; n=1, 2, ...).$$

Then the system (U, L_K, F_s) is complete with respect to (p_n). For

$$\sum_{i=1}^{\infty} p_i(x_i) = \sum_{i=1}^{\infty} \sup_{s \in B_i} \sum_{k=1}^{\infty} |a_k^{(i)} K_k(s)| < \infty,$$

$$x_i = (a_k^{(i)}) \in L_K \quad (i = 1, 2, ...)$$

implies that $\sum_{i=1}^{\infty} |a_k^{(i)}| < \infty$ if $K_k(s) \neq 0$ for at least one $s \in U$. Hence, taking $a_k = \sum_{i=1}^{\infty} a_k^{(i)}$ in this case, and $a_k = 0$ if $K_k(s) = 0$ for all $s \in U$, it follows by a simple computation that $x = (a_k) \in L_K$, where (4) and (5) are satisfied.

We deduce from Theorem 6

THEOREM 7. *If $K = (K_k)$, $K_k \in C(U)$ $(k = 1, 2, ...)$, then, for the system (U, L_K, F_s) with*

$$F_s(x) = \sum_{k=1}^{\infty} a_k K_k(s) \qquad (x = (a_k) \in L_K, s \in U),$$

the properties (A), $(\tilde{\text{A}})$, (A, I) *are equivalent, and each of these properties is equivalent to each of the statements* (a), (b):

(a) *the following conditions* (W_K) *and* (M_K) *are satisfied:*

(W_K)

$$\int_{B_n} K_k(t)\, d\alpha(t) = 0 \quad (k = 1, 2, ...), \qquad \int_{B_n} |d\alpha(t)| < \infty \tag{9}$$

imply $\alpha(t) = 0$ $(t \in B_n)$ for a normalized α on B_n for each $n = 1, 2, ...$;

(M_K) *for each $n = 1, 2, ...$ there exists an integer $i_n > n$ such that for all $i \geqslant i_n$*

$$\left| \sum_{k=1}^{m} a_k \int_{B_i} K_k(t)\, d\alpha(t) \right| \leqslant M \max_{s \in B_n} \sum_{k=1}^{m} |a_k K_k(s)|$$

$$(a_k \text{ complex}; m = 1, 2, ...), \qquad \int_{B_i} |d\alpha(t)| < \infty \tag{10}$$

imply $\alpha(t) = 0$ on all D_ν $(i_n \leqslant \nu \leqslant i)$ for a normalized α on B_i.

Furthermore, if for each $n = 1, 2, ...$ there is some $s_n \in B_n$ satisfying $\max_{s \in B_n} |K_k(s)| = |K_k(s_n)|$ *for all $k = 1, 2, ...$, then* (M_K) *is equivalent to the condition*

(M_K^0) *for each* $n=1,2,\ldots$ *there is an integer* $i_n > n$ *such that for all* $i \geqslant i_n$

$$\left|\int_{B_i} K_k(t)\,d\alpha(t)\right| \leqslant M\,|K_k(s_n)|\ (k=1,2,\ldots), \qquad \int_{B_i} |d\alpha(t)| < \infty \tag{11}$$

imply $\alpha(t)=0$ *on all* D_v $(i_n \leqslant v \leqslant i)$ *for a normalized* α *on* B_i.

(b) (W_K) *and the following property are satisfied: for each* $n=1,2,\ldots$ *there is an integer* $i_n > n$ *such that for any* $f \in C(B_i)$ $(i \geqslant i_n)$ *with* $f(s)=0$ $(s \in B_{i_n-1})$, *and any* $\varepsilon > 0$ *there is* $P(s)=\sum_{k=1}^m a_k K_k(s)$ *satisfying*

$$|f(s)-P(s)| < \varepsilon \quad (s \in B_i) \qquad \textit{and} \qquad \sum_{k=1}^m |a_k K_k(s)| < \varepsilon \quad (s \in B_n).$$

Theorem 7, which generalizes results of [5, 7], is an immediate consequence of Theorem 6, since $\int_{B_n} \sum_{k=1}^\infty a_k K_k(t)\,d\alpha(t)=0$ $((a_k)\in L_K)$ is equivalent to (9), and $|\int_{B_i} \sum_{k=1}^\infty a_k K_k(t)\,d\alpha(t)| \leqslant M \sup_{s\in B_n} \sum_{k=1}^\infty |a_k K_k(s)|$ $((a_k)\in L_K)$ is equivalent to (10) by reason of the uniform convergence of $\sum_{k=1}^\infty a_k K_k(s)$ $((a_k)\in L_K)$ on all B_i. Furthermore the equivalence of the conditions (b) of Theorem 6 and Theorem 7 in the case $X=L_K$ is obvious.

To deduce Theorem 2, and in particular the theorem of Eidelheit from Theorem 7 we set $D_i=[a_i,b_i]$, where $a_i=b_i=t_i$ $(i=1,2,\ldots)$ with $t_i \neq t_j$ $(i\neq j)$, and $K_k(t_i)=a_{ik}$ $(i,k=1,2,\ldots)$. Then $\int_{D_i} K_k(t)\,d\alpha(t)=\lambda_i a_{ik}$ with some constant λ_i, and therefore (L) and (W_K) are equivalent. Taking $a_i=0$ $(i\neq k)$, $a_k=1$ $(k=1,2,\ldots)$ it follows from (10) that

$$\left|\sum_{v=1}^i \lambda_v a_{vk}\right| \leqslant M \max_{1\leqslant v\leqslant n} |a_{vk}| \leqslant M \sum_{v=1}^n |a_{vk}| \qquad (k=1,2,\ldots).$$

On the other hand $|\sum_{v=1}^i \lambda_v a_{vk}| \leqslant M \sum_{v=1}^n |a_{vk}|$ implies

$$\left|\sum_{k=1}^m a_k \sum_{v=1}^i \lambda_v a_{vk}\right| \leqslant M \sum_{v=1}^n \sum_{k=1}^m |a_k a_{vk}| \leqslant nM \max_{1\leqslant v\leqslant n} \sum_{k=1}^m |a_k a_{vk}|$$

for all a_k; i.e., the inequality (10) with the constant nM. Thus (N) and (M_K) are equivalent for (a_{ik}), where the equivalence of the conditions (b) of Theorem 2 and Theorem 7 in this case is obvious. This proves Theorem 2.

4. Applications to Dirichlet Series

If $K_k(s)=s^{\lambda_k}$ on $U=[0,1)$ or on $U=[0,\infty)$, Theorem 7 has applications to asymptotic approximation and interpolation by Dirichlet series. Referring to this we say that a sequence (λ_k) of exponents $\lambda_k \geq 0$ $(k=0, 1, 2, ...)$ has the property (M_d) with respect to a number $d \in (0, 1]$ if

$$\int_0^1 t^{\lambda_k}\, d\alpha(t) = \mathcal{O}(q^{\lambda_k} d^{\lambda_k}) \quad (k \to \infty), \qquad \int_0^1 |d\alpha(t)| < \infty$$

imply $\alpha(t)=0$ on $(q, 1]$ for each $q \in (0, 1)$, where α is normalized. It is obvious that (M_1) implies (M_d) for $d \in (0, 1)$, and (M_c) follows from (M_d) for $0 < c < d \leq 1$.

The following result, which also is used in the proofs of Theorem 3 and Theorem 5, is due to [8].

THEOREM 8. *If* $\lambda_k \geq 0$, $\lambda_{k+1} - \lambda_k \geq c > 0$ $(k=0, 1, 2, ...)$, *and* $\sum_{k=1}^{\infty} \lambda_k^{-1} = \infty$, *then* (λ_k) *has the property* (M_1).

An alternative proof of Theorem 8 is given in [4, Corollary of Theorem 3]. Concerning Dirichlet series we say that (λ_k) has the property $(A_{[0,1)})$ if for every $f, h \in C[0, 1)$, $h(s) > 0$ $(s \in [0, 1))$, there is an absolutely converging series $g(s) = \sum_{k=0}^{\infty} a_k s^{\lambda_k}$ $(s \in [0, 1))$ such that

$$|f(s) - g(s)| < h(s) \qquad (s \in [0, 1)),$$

where we use corresponding definitions for $(A_{[0,1)}, I)$, $(A_{[0,\infty)})$, $(A_{[0,\infty)}, I)$, respectively.

Without proof we state the following result, which can be deduced from Theorem 7.

THEOREM 9. *A sequence* (λ_k), $0 = \lambda_0 < \lambda_k$ $(k=1, 2, ...)$ *has the property* $(A_{[0,1)})$ *if and only if* (λ_k) *has the property* $(A_{[0,1)}, I)$, *and if and only if* (M_1) *is satisfied. A sequence* (λ_k), $0 = \lambda_0 < \lambda_k$ $(k=1, 2, ...)$ *has the property* $(A_{[0,\infty)})$ *if and only if* (λ_k) *has the property* $(A_{[0,\infty)}, I)$, *and if and only if* (M_d) *is satisfied for some* $d \in (0, 1]$.

Concerning $(A_{[0,1)})$ and $(A_{[0,\infty)})$ Theorem 9 was proved first in [4, 5], and the part for simultaneous approximation and interpolation is due to Metz [7].

5. PROOFS OF THEOREM 3 AND THEOREM 5

The deduction of these theorems from Theorem 6 and 7 is based on Theorem 8.

Proof of Theorem 3. We first assume (3). Taking $K_k(s) = s^{m_{k-1}}$ $(k=1, 2, ...)$, $D_{2\nu-1} = [\nu - 1, \nu]$, $D_{2\nu} = [-\nu, -\nu+1]$ $(\nu = 1, 2, ...)$, and so

$B_{2i}=[-i, i]$ $(i=1, 2, ...)$, $B_{2i+1}=[-i, i+1]$ $(i=0, 1, 2, ...)$, $U=(-\infty, \infty)$, we have

$$\int_{B_{2i}} t^{m_k}\, d\alpha(t) = \int_0^i t^{m_k}[d\alpha(t) + e_k\, d\alpha(-t)] \qquad (i=1, 2, ...), \tag{12}$$

$$\int_{B_{2i+1}} t^{m_k}\, d\alpha(t) = \int_0^i t^{m_k}[d\alpha(t) + e_k\, d\alpha(-t)] + \int_i^{i+1} t^{m_k}\, d\alpha(t) \qquad (i=0, 1, 2, ...), \tag{13}$$

where $e_k=1$ (m_k even), $e_k=-1$ (m_k odd). Applying Theorem 8 it follows from (12) and (13) by an easy computation that the conditions ($\mathrm{W_K}$) and (M_k^0) of Theorem 7 are satisfied. Thus we obtain (A, I). Conversely, if (A) is supposed, we can find for every $w \in C[0, 1]$ and $\varepsilon > 0$ an entire g with $g(s)=\sum_{k=0}^{\infty} a_k s^{m_k}$ $(-\infty < s < \infty)$ such that in particular $g(s)=w(s)+\varepsilon(s)$, $g(-s)=w(s)+\varepsilon(-s)$ with $|\varepsilon(s)|$, $|\varepsilon(-s)| < \varepsilon$ for $s \in [0, 1]$. Taking $b(s)=2^{-1}[g(s)+g(-s)]$ we have

$$b(s) = \sum_{\substack{k \geqslant 0 \\ m_k \text{ even}}} a_k s^{m_k} = w(s) + 2^{-1}[\varepsilon(s)+\varepsilon(-s)],$$

$$|w(s)-b(s)| < \varepsilon \qquad (s \in [0, 1]). \tag{14}$$

Since the series of $b(s)$ converges uniformly on $[0, 1]$ we can replace $b(s)$ in (14) by a polynomial $\sum_{0 \leqslant k \leqslant N,\, m_k \text{ even}} a_k s^{m_k}$. Thus we obtain $\sum_{k \geqslant 1,\, m_k \text{ even}} m_k^{-1} = \infty$ by the theorem of Müntz [11, p. 336]. A similar argument proves $\sum_{k \geqslant 1,\, m_k \text{ odd}} m_k^{-1} = \infty$, which completes the proof of Theorem 3.

Proof of Theorem 5. We deduce Theorem 5 from Theorem 6 by choosing $X=G$ to be the class of all entire functions g such that $\int_0^\infty |g(u)|\, u^s\, du < \infty$ $(s \geqslant 0)$. We set $F_s(g)=\int_0^\infty g(u) u^s\, du$ $(g \in G)$, $p_n(g) = \int_0^\infty |g(u)|\, u^n\, du + \max_{|z|=n} |g(z)|$ $(n = 1, 2, ...;$ z complex$)$, and $D_i = [i-1, i]$, $B_i=[0, i]$ $(i=1, 2, ...)$, $U=[0, \infty)$. Then the completeness of the system (U, G, F_s) follows at once. We take, in particular, $g_k \in G$ with $g_k(u)=u^k e^{-u}$ $(k=0, 1, 2, ...)$, and verify (W) and (M) of Theorem 6 by proving (δ) and (γ):

(δ)

$$\int_0^n \int_0^\infty g_k(u) u^t\, du\, d\alpha(t) = \int_0^n \Gamma(k+t+1)\, d\alpha(t) = 0$$

$$(k=0, 1, 2, ...), \quad \int_0^n |d\alpha(t)| < \infty \tag{15}$$

imply $\alpha(t)=0$ on $[0, n]$ for each $n=1, 2, \ldots$;

(γ)

$$\left|\int_0^i \Gamma(k+t+1)\, d\alpha(t)\right| \leqslant Mp_n(g_k) = M[\Gamma(k+n+1)+e^n n^k] \qquad (k=0, 1, 2, \ldots),$$

$$\int_0^i |d\alpha(t)| < \infty \tag{16}$$

imply $\alpha(t)=0$ on $(n, i]$ for all $i>n$; $n=1, 2, \ldots$, where α is normalized.

By Stirling's formula

$$\log \Gamma(y) = (y-2^{-1})\log y - y + \log\sqrt{2\pi} + \mathcal{O}(y^{-1}) \quad (y\to\infty)$$

a simple calculation gives

$$\Gamma(k+t+1) = \Gamma(k+1)\, e^{t\log k}[1+r_k(t)], \tag{17}$$

and

$$|r_k(t)| \leqslant b_n k^{-1} \qquad (t\in[0, n];\ k=0, 1, 2, \ldots), \tag{18}$$

where the constant b_n only depends on n. Hence (15), (17), and (18) imply

$$\int_0^n e^{t\log k}\, d\alpha(t) = -\int_0^n e^{t\log k} r_k(t)\, d\alpha(t) = \mathcal{O}(e^{(n-1)\log k}) \qquad (k\to\infty)$$

and so, by taking $k=2^v$,

$$\int_0^n e^{tv\log 2}\, d\alpha(t) = \mathcal{O}(e^{(n-1)v\log 2}) \qquad (v\to\infty). \tag{19}$$

Hence, using a simple substitution, it follows from Theorem 8 that $\alpha(t)=0$ on $(n-1, n]$. Repeating this argument n times we obtain $\alpha(t)=0$ on $(0, n]$, and so $\alpha(t)=0$ on $[0, n]$ by (15), since $\Gamma(1)\neq 0$ and α is normalized. This is (δ). To prove (γ), we conclude from (16), (17), and (18) that

$$\begin{aligned}\int_0^i e^{t\log k}\, d\alpha(t) &= -\int_0^i e^{t\log k} r_k(t)\, d\alpha(t) \\ &\quad + \mathcal{O}(\Gamma(k+n+1)[\Gamma(k+1)]^{-1}) + \mathcal{O}(n^k[\Gamma(k+1)]^{-1}) \\ &= \mathcal{O}(e^{(i-1)\log k}) + \mathcal{O}(e^{n\log k}) + \mathcal{O}(1) \\ &= \mathcal{O}(e^{(i-1)\log k}) \qquad (k\to\infty),\end{aligned}$$

which, by Theorem 8, implies $\alpha(t)=0$ on $(i-1, i]$. Hence, repeating this argument, we obtain $\alpha(t)=0$ on $(n, i]$. This completes the proof of Theorem 5.

6. An Identity Theorem of Functional Analysis

Next, we prove an identity theorem of functional analysis which will play an essential part in the proof of our main Theorem 6.

Theorem 10. *Suppose that X is a linear space with a seminorm p on X, and that A is a linear mapping from X into the normed linear space Y with a norm $\|y\|$ for $y \in Y$. If $G \subseteq Y$, then, in order that for every $y \in G$ and $\varepsilon > 0$ there exist an element $x \in X$ satisfying*

$$\|y - A(x)\| < \varepsilon \qquad \text{and} \qquad p(x) < \varepsilon, \tag{20}$$

the condition $\mathrm{I}_G(X, Y, A, p)$ is necessary and sufficient, where $\mathrm{I}_G(X, Y, A, p)$ denotes the identity property that

$$|F(A(x))| \leqslant Mp(x) \qquad (x \in X),$$

for a bounded linear functional F on Y, always implies $F(y)=0$ $(y \in G)$.

Proof of Theorem 10. We assume first $\mathrm{I}_G(X, Y, A, p)$. Let X/N denote the quotient space of all sets $\pi(x)=\{x+t : t \in N\}$ $(x \in X)$, where $N=\{x \in X : p(x)=0\}$. Then X/N is a normed linear space with the norm $\tilde{p}$ by taking $\tilde{p}(\pi(x))=p(x)$ $(x \in X)$ [12, p. 31]. If T is the space of all ordered pairs $w=(y, \pi(x))$ $(y \in Y, x \in X)$, then

$$\|w\| = \|(y, \pi(x))\| = \|y\| + p(x) \tag{21}$$

defines a norm on the linear space T. Let (y_0, x_0) denote the null element of T, where y_0, x_0 are the null elements of Y and X/N respectively. The set V of all pairs $(A(x), \pi(x))$ $(x \in X)$ is a linear subspace of T. Suppose that F is a bounded linear functional on T satisfying $F(w)=0$ $(w \in V)$. Then, to prove (20), it is enough to show $F(y, x_0)=0$ $(y \in G)$ [11, p. 114]. We have

$$F(y, \pi(x)) = F(y, x_0) + F(y_0, \pi(x)) \qquad (y \in Y, x \in X), \tag{22}$$

where $F(y, x_0)$, $F(y_0, \pi(x))$ define bounded linear functionals on Y and X/N, respectively. Thus $F(A(x), \pi(x))=0$ $(x \in X)$ and (22) imply $F(A(x), x_0) = -F(y_0, \pi(x))$ $(x \in X)$, and therefore, it follows from (21) that $|F(A(x), x_0)| = |F(y_0, \pi(x))| \leqslant \|F\|\, p(x)$ $(x \in X)$, where $\|F\|$ denotes the norm of the functional F. Hence we obtain $F(y, x_0)=0$ $(y \in G)$ by

$I_G(X, Y, A, p)$. Conversely, we assume that (20) can be satisfied for any $y \in G$ and $\varepsilon > 0$, and assume that

$$|F(A(x))| \leqslant Mp(x) \qquad (x \in X) \tag{23}$$

for a bounded linear functional F on Y. Hence, if $F(b) \neq 0$ for some $b \in G$, we can find $x_k \in X$ $(k = 1, 2, ...)$ such that

$$F(A(x_k)) \to F(b) \neq 0 \qquad (k \to \infty) \tag{24}$$

and

$$p(x_k) \to 0 \qquad (k \to \infty). \tag{25}$$

Thus (23) and (25) imply $|F(A(x_k))| \leqslant Mp(x_k) \to 0$ $(k \to \infty)$ in contradiction to (24). This completes the proof of Theorem 10.

7. The Proof of Theorem 6

We state the following

Lemma. (a) *The condition* (W) *of a system* (X, U, F_s) *is equivalent to the following property: For each* $f \in C(B_n)$, $B_n = \bigcup_{i=1}^n D_i$ $(n = 1, 2, ...)$, *each* $\varepsilon > 0$, *and for arbitrary numbers* $q_\nu \in B_n$ $(\nu = 1, ..., N)$ *there is an element* $x \in X$ *such that*

$$|f(s) - F_s(x)| < \varepsilon \quad (s \in B_n) \qquad \textit{and} \qquad F_{q_\nu}(x) = f(q_\nu) \quad (\nu = 1, ..., N).$$

(b) *The conditions* $(A_{p,m,j})$ *and* $(A^{I}_{p,m,j})$ *concerning a seminorm p are equivalent for all integers m, j with* $1 < m \leqslant j$.

Proof. We first prove (b). Obviously $(A^{I}_{p,m,j})$ implies $(A_{p,m,j})$, and we now assume $(A_{p,m,j})$ for fixed integers $1 < m \leqslant j$. Suppose that $f \in C(B_j)$ with $f(s) = 0$ $(s \in B_{m-1})$, and that $\varepsilon > 0$. To prove $(A^{I}_{p,m,j})$ we have to show by induction on N that for any different numbers $q_\nu \in B_j$ $(\nu = 1, ..., N)$ there is an element $x \in X$ satisfying

$$|f(s) - F_s(x)| < \varepsilon \quad (s \in B_j), \quad p(x) < \varepsilon, \qquad \text{and} \qquad F_{q_\nu}(x) = f(q_\nu) \quad (\nu = 1, ..., N). \tag{26}$$

If $N = 1$, then by $(A_{p,m,j})$ we find $x_1 \in X$ such that $|f(s) - F_s(x_1)| < \varepsilon$ $(s \in B_j)$ and $p(x_1) < \varepsilon$, and in particular

$$F_{q_1}(x_1) = f(q_1) + d_1, \qquad |d_1| < \varepsilon, \tag{27}$$

where we may assume $\varepsilon < |f(q_1)|$ if $f(q_1) \neq 0$. If $d_1 = 0$, then our conclusion is trivially true for $N = 1$. Suppose therefore that $d_1 \neq 0$. Then $F_{q_1}(x_1) = f(q_1) + d_1 \neq 0$, which follows from $d_1 \neq 0$ in the case $f(q_1) = 0$, and which is a consequence of $|d_1| < \varepsilon < |f(q_1)|$ if $f(q_1) \neq 0$. We choose $\varepsilon_1 > 0$ satisfying

$$\varepsilon_1 < 2^{-1}\varepsilon, \qquad \varepsilon_1 \, |F_{q_1}(x_1)|^{-1} \, |F_s(x_1)| < 2^{-1}\varepsilon \quad (s \in B_j),$$

and (28)

$$\varepsilon_1 \, |F_{q_1}(x_1)|^{-1} \, p(x_1) < 2^{-1}\varepsilon,$$

and by $(A_{p,m,j})$ we can find $x_2 \in X$ such that

$$|f(s) - F_s(x_2)| < \varepsilon_1 \quad (s \in B_j) \qquad \text{and} \qquad p(x_2) < \varepsilon_1 \tag{29}$$

and therefore

$$F_{q_1}(x_2) = f(q_1) + b_1, \qquad |b_1| < \varepsilon_1. \tag{30}$$

Taking $x = x_2 - b_1[F_{q_1}(x_1)]^{-1} x_1$ we obtain

$$F_s(x) = F_s(x_2) - b_1[F_{q_1}(x_1)]^{-1} F_s(x_1) \tag{31}$$

and therefore, it follows from (27)–(31) that

$$|f(s) - F_s(x)| \leqslant |f(s) - F_s(x_2)|$$
$$+ \varepsilon_1 \, |F_{q_1}(x_1)|^{-1} \, |F_s(x_1)| < \varepsilon \qquad (s \in B_j),$$
$$p(x) \leqslant p(x_2) + |b_1| \, |F_{q_1}(x_1)|^{-1} \, p(x_1) < \varepsilon,$$

and

$$F_{q_1}(x) = f(q_1),$$

which proves (b) for $N = 1$. We now assume that our conclusion (26) is true for N. If $f(q_\nu) = 0$ for at least one of the given numbers $q_\nu \in B_j$ $(\nu = 1, ..., N+1)$, then we choose the denotation of q_ν such that $f(q_\nu) = 0$ if and only if $1 \leqslant \nu \leqslant n_0 \leqslant N+1$. By (26) we determine $x_1 \in X$ satisfying

$$|f(s) - F_s(x_1)| < \varepsilon \quad (s \in B_j), \qquad p(x_1) < \varepsilon,$$

and (32)

$$F_{q_\nu}(x_1) = f(q_\nu) \quad (\nu = 1, ..., N),$$

and therefore

$$F_{q_{N+1}}(x_1)=f(q_{N+1})+d_1, \qquad |d_1|<\varepsilon. \tag{33}$$

Obviously (26) is satisfied for $N+1$ if $d_1=0$, and we suppose $d_1\neq 0$. Next, we show that there is an element $x_0\in X$ such that

$$F_{q_\nu}(x_0)=0 \quad (\nu=1, ..., N), \qquad \text{and} \qquad F_{q_{N+1}}(x_0)\neq 0. \tag{34}$$

If $f(q_{N+1})=0$, and so $f(q_\nu)=0$ $(\nu=1, ..., N)$ by our denotation of the q_ν, we obtain (34) immediately from (32) and (33) by taking $x_0=x_1$, where $F_{q_{N+1}}(x_0)=d_1\neq 0$. But in the case $f(q_{N+1})\neq 0$ we can find $x_0\in X$ to satisfy (34) by applying our assumption (26) for N to a function $w\in C(B_j)$, which is chosen such that $w(s)=0$ $(s\in B_{m-1})$, $w(q_\nu)=0$ $(\nu=1, ..., N)$, and $w(q_{N+1})=f(q_{N+1})\neq 0$. If

$$0<\varepsilon_1<2^{-1}\varepsilon, \qquad \varepsilon_1\,|F_{q_{N+1}}(x_0)|^{-1}\,|F_s(x_0)|<2^{-1}\varepsilon \quad (s\in B_j),$$

and (35)

$$\varepsilon_1\,|F_{q_{N+1}}(x_0)|^{-1}\,p(x_0)<2^{-1}\varepsilon,$$

there is by (26) an element $x_2\in X$ such that

$$|f(s)-F_s(x_2)|<\varepsilon_1 \quad (s\in B_j), \qquad p(x_2)<\varepsilon_1,$$

and (36)

$$F_{q_\nu}(x_2)=f(q_\nu) \quad (\nu=1, ..., N),$$

and in particular

$$F_{q_{N+1}}(x_2)=f(q_{N+1})+b_1, \qquad |b_1|<\varepsilon_1. \tag{37}$$

Taking $x=x_2-b_1[F_{q_{N+1}}(x_0)]^{-1}x_0$, and so

$$F_s(x)=F_s(x_2)-b_1[F_{q_{N+1}}(x_0)]^{-1}F_s(x_0), \tag{38}$$

it follows from (34)–(38) that

$$\begin{aligned} |f(s)-F_s(x)| &\leqslant |f(s)-F_s(x_2)| \\ &\quad +\varepsilon_1\,|F_{q_{N+1}}(x_0)|^{-1}\,|F_s(x_0)|<\varepsilon \qquad (s\in B_j), \\ p(x) &\leqslant p(x_2)+|b_1|\,|F_{q_{N+1}}(x_0)|^{-1}\,p(x_0)<\varepsilon, \end{aligned}$$

and finally $F_{q_\nu}(x)=f(q_\nu)$ $(\nu=1, ..., N+1)$. This completes the proof of (b). It remains to prove (a). By a well known result [11, p. 114], the condition

(W) is equivalent to the property that for each $f \in C(B_n)$ and each $\varepsilon > 0$ there is $x \in X$ such that

$$|f(s) - F_s(x)| < \varepsilon \qquad (s \in B_n). \tag{39}$$

The proof, by induction on N, that we can choose $x \in X$ to satisfy, in addition to (39), the equations $F_{q_\nu}(x) = f(q_\nu)$ $(\nu = 1, ..., N)$ is a simple variant of the proof of (b). Our lemma is thus established.

Proof of Theorem 6. Obviously (A, I) implies (A), and ($\tilde{\mathrm{A}}$) is a consequence of (A). We suppose first ($\tilde{\mathrm{A}}$), and prove (W) and (M). It follows immediately from ($\tilde{\mathrm{A}}$) that we have $|f(s) - F_s(x)| < \varepsilon$ $(s \in B_n)$ for each $n = 1, 2, ...$, each $f \in C(B_n)$, and any $\varepsilon > 0$ by choosing an appropriate $x \in X$. Thus we obtain (W) [11, p. 114].

We now assume that (M) is not satisfied. Then by (6) we can find an integer $n \geq 1$ and increasing integers i_l with $n < i_l < i_{l+1}$ $(l = 1, 2, ...)$ such that there are normalized functions α_l on $B_{i_{l+1}} = \bigcup_{\nu=1}^{i_{l+1}} D_\nu$ and closed intervals $E_l \subseteq \bigcup_{\nu=i_l}^{i_{l+1}} D_\nu$ satisfying the conditions

$$\left| \int_{B_{i_{l+1}}} F_t(x)\, d\alpha_l(t) \right| \leq M_l\, p_n(x) \quad (x \in X),$$
$$\int_{B_{i_{l+1}}} |d\alpha_l(t)| < \infty \quad (l = 1, 2, ...), \tag{40}$$

where the constants M_l are independent of x, and

$$\int_{E_l} d\alpha_l(t) \neq 0 \qquad (l = 1, 2, ...). \tag{41}$$

Moreover we can choose these i_l and E_l with $E_{l+1} \cap B_{i_{l+1}} = \varnothing$ $(l = 1, 2, ...)$. Thus the sets E_l $(l = 1, 2, ...)$ are disjoint, and

$$E_j \cap B_{i_{l+1}} = \varnothing \qquad (j \geq l+1;\, l = 1, 2, ...). \tag{42}$$

Multiplying (40) by constants we may assume that

$$\left| \int_{B_{i_{l+1}}} F_t(x)\, d\alpha_l(t) \right| \leq p_n(x) \qquad (x \in X;\, l = 1, 2, ...) \tag{43}$$

and

$$\int_{B_{i_{l+1}}} |d\alpha_l(t)| \leq 1 \qquad (l = 1, 2, ...). \tag{44}$$

Proceeding successively, by (41) we can find functions $w_l \in C(U)$ $(l=1, 2, ...)$ such that

$$\left| \int_{B_{i_{l+1}}} w_l(t)\, d\alpha_l(t) \right| > l + \sum_{\nu=1}^{l-1} \int_{B_{i_{l+1}}} |w_\nu(t)|\, |d\alpha_l(t)| \qquad (l=2, 3, ...), \quad (45)$$

if we construct w_l with respect to the properties of the intervals D_i as follows: We choose $w_l(t)$ to be suitable large constants on each of the disjoint E_l, and we set $w_l(t)=0$ on all D_i with $E_l \cap D_i = \emptyset$. Thus by (42), in particular

$$w_j(t)=0 \qquad (t \in B_{i_{l+1}}; j \geqslant l+1; l=1, 2, ...). \quad (46)$$

If $E_l \cap D_i \neq \emptyset$, and $E_l = [c_l, d_l] \neq D_i$, i.e., $[c_l - \varepsilon, c_l] \subseteq D_i$ or $[d_l, d_l + \varepsilon] \subseteq D_i$ for some $\varepsilon > 0$, then we set $w_l(t)=0$ $(t \leqslant c_l - \varepsilon)$ or $w_l(t)=0$ $(t \geqslant d_l + \varepsilon)$, and choose w_l to be linear on $[c_l - \varepsilon, c_l]$ or $[d_l, d_l + \varepsilon]$.

Since $\int_{c_l-\varepsilon}^{c_l-0} |d\alpha(t)|$, $\int_{d_l+0}^{d_l+\varepsilon} |d\alpha(t)| \to 0$ $(\varepsilon \to +0)$, if α is of bounded variation, we obtain (45) for sufficiently small ε. Let

$$f(t) = \sum_{j=1}^{\infty} w_j(t) \qquad (t \in U). \quad (47)$$

Then, by (46), the series (47) converges absolutely and uniformly on each B_i $(i=1, 2, ...)$. This implies $f \in C(U)$. Hence, by (45), (46), and (47)

$$\begin{aligned} \left| \int_{B_{i_{l+1}}} f(t)\, d\alpha_l(t) \right| &= \left| S_{B_{i_{l+1}}} \sum_{j=1}^{l} w_j(t)\, d\alpha_l(t) \right| \\ &\geqslant \left| \int_{B_{i_{l+1}}} w_l(t)\, d\alpha_l(t) \right| \\ &\quad - \sum_{j=1}^{l-1} \int_{B_{i_{l+1}}} |w_j(t)|\, |d\alpha_l(t)| > l \qquad (l=2, 3, ...). \end{aligned} \quad (48)$$

By $(\tilde{A})$ there is $x_0 \in X$ satisfying $|f(t) - F_t(x_0)| < 1$ $(t \in U)$. Thus it follows from (44) and (48) that

$$\left| \int_{B_{i_{l+1}}} F_t(x_0)\, d\alpha_l(t) \right| > l - 1 \qquad (l=2, 3, ...). \quad (49)$$

On the other hand, by (43), we have

$$\left| \int_{B_{i_{l+1}}} F_t(x_0)\, d\alpha_l(t) \right| \leqslant p_n(x_0) \qquad (l=1, 2, ...),$$

which contradicts (49). This proves (M).

We now assume (W) and (M). Taking X with the seminorm $p(x)=p_n(x)$ for each n, $Y=C(B_i)$ $(i\geqslant i_n)$ with the norm $\|f\|=\max_{t\in B_i}|f(t)|$ for $f\in Y$, $A(x)=F_t(x)$ $(x\in X)$, and $G\subseteq Y$ to be the class of all $f\in C(B_i)$ with $f(t)=0$ $(t\in B_{i_n-1})$, it follows from Theorem (10), by the Riesz representation theorem [11, p. 139], that (M) is equivalent to the existence of an integer $i_n>n$ for each n satisfying $(A_{p_n,i_n,i})$ for all $i\geqslant i_n$. Our above lemma, (b), asserts that $(A_{p_n,i_n,i})$ is equivalent to $(A^{I}_{p_n,i_n,i})$. Thus the statements (a), (b), and (c) of Theorem 6 are equivalent. It remains to prove (A, I). In particular it follows from (M) that for each $n=1,2,...$ there is an $i_n>n$ with $i_{n+1}>i_n$ such that $(A^{I}_{p_n,i_n,i_{n+1}-1})$ is satisfied. Suppose $f, h\in C(U)$, $h(s)>0$ $(s\in U)$, $U=\bigcup_{i=1}^{\infty}D_i$, $q_v\in U$ $(v=1,2,...)$, where each D_i only contains a finite number of these q_v. Let

$$\varepsilon_n=\min_{s\in V_n}h(s)>0,\quad V_n=\bigcup_{i=i_n}^{i_{n+1}-1}D_i,\quad V_0=B_{i_1-1}\qquad(n=0,1,2,...).$$

We may assume that

$$\varepsilon_{n+1}<2^{-1}\varepsilon_n\qquad(n=0,1,2,...).\tag{50}$$

Next, we successively determine elements $x_n\in X$ $(n=1,2,...)$ as follows: By (W) and our lemma, (a), we choose $x_1\in X$ such that

$$|f(s)-F_s(x_1)|<2^{-1}\varepsilon_0\qquad(s\in V_0)\tag{51}$$

and

$$F_s(x_1)=f(s)\qquad\text{for all } s=q_v\in V_0 \text{ and for all frontier points } s \text{ of } V_0.$$

Suppose $x_i\in X$ $(i=1,...,n-1)$ have already been determined with the property that

$$\sum_{i=1}^{n-1}F_s(x_i)=f(s)\qquad\text{for all } s=q_v\in B_{i_n-1} \text{ and for all frontier points } s \text{ of } B_{i_n-1}.\tag{52}$$

Then we set $h(s)=0$ $(s\in B_{i_n-1})$, $h(s)=f(s)-\sum_{i=1}^{n-1}F_s(x_i)$ $(s\in V_n)$. Thus $h\in C(B_{i_{n+1}-1})$, and by $(A^{I}_{p_n,i_n,i_{n+1}-1})$ we can choose $x_n\in X$, and so $F_s(x_n)$, to approximate and interpolate $h(s)$ such that

$$\left|f(s)-\sum_{i=1}^{n}F_s(x_i)\right|<2^{-1}\varepsilon_n\qquad(s\in V_n),\tag{53}$$

$$|F_s(x_n)|<2^{-1}\varepsilon_n\qquad(s\in B_{i_n-1}),\tag{54}$$

$$p_n(x_n)<2^{-n},\tag{55}$$

and finally,

$$F_s(x_n) = h(s) \qquad \text{for all } s = q_v \in B_{i_{n+1}-1} \text{ and for all frontier points } s \text{ of } B_{i_{n+1}-1}. \tag{56}$$

It follows from (52) and (56) that

$$\sum_{i=1}^{n} F_s(x_i) = f(s) \qquad \text{for all } s = q_v \in B_{i_{n+1}-1} \text{ and for all frontier points } s \text{ of } B_{i_{n+1}-1} \quad (n = 1, 2, ...), \tag{57}$$

and, in particular, that

$$F_{q_v}(x_j) = 0 \qquad \text{for all} \quad q_v \in B_{i_n-1} \quad (j \geqslant n; n = 1, 2, ...). \tag{58}$$

We set

$$g(s) = \sum_{i=1}^{\infty} F_s(x_i). \tag{59}$$

By (55) we have $\sum_{i=1}^{\infty} p_i(x_i) < \infty$. Thus, by the completeness of (U, X, F_s) the series (59) converges absolutely on U, and there is an element $x \in X$ such that $g(s) = F_s(x)$ $(s \in U)$.

If $s \in U$, then $s \in V_n$ for some $n = 0, 1, 2, ...$, and therefore it follows from (50), (51), (53), (54), and (59) that

$$\begin{aligned} |f(s) - g(s)| &\leqslant \left| f(s) - \sum_{i=1}^{n} F_s(x_i) \right| + \sum_{i=n+1}^{\infty} |F_s(x_i)| \\ &< 2^{-1}\varepsilon_n + 2^{-1} \sum_{i=n+1}^{\infty} \varepsilon_i \\ &< 2^{-1}\varepsilon_n + 2^{-1}\varepsilon_n \sum_{i=1}^{\infty} 2^{-i} \\ &= \varepsilon_n \leqslant h(s) \qquad (s \in V_n), \end{aligned}$$

which proves (A). Moreover, if $q_v \in B_{i_n-1}$ $(n = 1, 2, ...)$, we have by (57), (58), and (59) that

$$F_{q_v}(x) = \sum_{i=1}^{n} F_{q_v}(x_i) + \sum_{i=n+1}^{\infty} F_{q_v}(x_i) = f(q_v).$$

This proves (A, I) and completes the proof of Theorem 6.

We might mention that we have proved the necessity of (W) and (M) for $(\tilde{A})$ without use of the completeness of the system (U, X, F_s).

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